BP: MMATH LINEAR ALGEBRA

All vector spaces considered below are assumed to be finite dimensional.

(1) (14+14=28 points) Answer the following multiple choice questions about each of them. Justify your answer.

(a) Let R be a ring and $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Let M be a R-module. The sequence

$$0 \to Hom_R(M, A) \to Hom_R(M, B) \to Hom_R(M, C) \to 0$$

is exact if

(i) M is a projective R-module.
(ii) A is a projective R-module.
(iii) B is a projective R-module.
(iv) C is a projective R-module.

(b) Let R be a principal ideal domain. Which of the following are true.

(i) Every submodule of a free R-module is torsion free.
(ii) Every torsion free submodule of a finitely generated R-module is free.
(iii) Every torsion free R-module is free.
(iv) Every ideal of R is a free R-module.

(2) (8+8+8+8=32 points) Prove or disprove (using a counterexample) the following statements.

(a) Let ϕ and ψ be two commuting diagonalizable linear operators on a finite dimensional vector space V. Then ϕ and ψ are simultaneously diagonalizable.

(b) Let A and B be $n \times n$ matrices such that minimal polynomial of A and B are equal and has degree n. Then A and B are similar matrices.

(c) Let R be an integral domain and M be a finitely generated torsion free R-module then M is a free R-module.

(d) Let R be a commutative and M be an R-module. Let $\phi : M \to M$ be an R-linear map. Then ϕ extends to a R-algebra homomorphism from the tensor algebra $T(M)$ to the symmetric algebra $S(M)$.

(3) (15 points) Let A be a $n \times n$ hermitian matrix. Then show that $\exp(A)$ is a non-negative operator on the hermitian space $\mathbb{C}^n$ with the standard hermitian form.

(4) (15 points) Compute and list down rational canonical form and Jordan form of all the 4×4 real matrices whose minimal polynomial is $x^2 - 9$.

(5) (10 points) Let R be an integral domain and M be an R-module. Let $m_1, \ldots, m_n$ be generators of M as an R-module. Suppose there exist elements $\phi_1, \ldots, \phi_n \in Hom_R(M, R)$ such that $\phi_i(m_i) \neq 0$ for all i and $\phi_i(m_j) = 0$ for all $i \neq j$. Show that $(m_1, \ldots m_n)$ is a basis of M and $(\phi_1, \ldots, \phi_n)$ is a linearly independent subset of $Hom_R(M, R)$.